

An Incidence Bound for k -planes in F^n and a Planar Variant of the Kakeya Maximal Function

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Abstract

We discuss a planar variant of the Kakeya maximal function in the setting of a vector space over a finite field. Using methods from incidence combinatorics, we demonstrate that the operator is bounded from L^p to L^q when $1 \leq p \leq \frac{kn+k+1}{k(k+1)}$ and $1 \leq q \leq (n-k)p'$.

1 Introduction

The Kakeya conjecture is a long standing open problem in the field of geometric combinatorics which is concerned with the extent to which a large direction-separated family of thin tubes can be compressed into a small space. There are essentially two formulations of the conjecture: one geometric and the other analytic. In order to state the geometric formulation of the conjecture, we shall need the following fundamental definition:

Definition 1.1. A set $E \subset \mathbb{R}^n$ is said to be a *Kakeya set* if for any direction $\xi \in S^{n-1}$, there exists a unit line segment l_ξ parallel to ξ such that $l_\xi \subset E$.

The Kakeya conjecture is concerned with the dimension of such objects. Explicitly,

Conjecture 1.2. If $E \subset \mathbb{R}^n$ is a Kakeya set, then $\dim(E) = n$.

Strictly speaking, Conjecture 1.2 should be interpreted as three separate conjectures, as one can consider the Hausdorff dimension (denoted $\dim_H(E)$), lower Minkowski dimension (denoted $\underline{\dim}(E)$) or upper Minkowski dimension (denoted $\overline{\dim}(E)$) of such sets; these different notions of dimension are discussed

at great length in most texts on geometric measure theory, for example [9] and [14]. By definition, one has

$$\dim_H(E) \leq \underline{\dim}(E) \leq \overline{\dim}(E)$$

for any set $E \subset \mathbb{R}^n$. Therefore, any progress on the Hausdorff version of the Kakeya conjecture will immediately imply progress on the other two versions. It is not known, however, whether all three versions of the conjecture are actually equivalent. Furthermore, the best known results concerning the conjecture differ for each of the various notions of dimension (see [10] and [11] for examples of this phenomenon).

The analytic formulation of the conjecture concerns the boundedness of the Kakeya maximal function:

Definition 1.3. Given a function $f \in L^1_{loc}(\mathbb{R}^n)$ and a number $0 < \delta \ll 1$, we may define the *Kakeya maximal function* as

$$f_\delta^* : S^{n-1} \rightarrow \mathbb{R}$$

$$f_\delta^*(\xi) = \sup_{a \in \mathbb{R}^n} \frac{1}{|T_\delta(a, \xi)|} \int_{T_\delta(a, \xi)} |f(x)| dx$$

where $T_\delta(a, \xi)$ denotes the tube of dimensions $1 \times \delta^{n-1}$ centered at the point a , oriented in the direction ξ .

With this definition, we may state the analytic version of the conjecture:

Conjecture 1.4. For all $\epsilon > 0$, and $0 < \delta \ll 1$, one has

$$\|f_\delta^*\|_{L^p(S^{n-1}, d\sigma)} \leq C_\epsilon \delta^{\frac{n}{p}-1-\epsilon} \|f\|_{L^p(\mathbb{R}^n, dx)} \quad (1)$$

for $1 \leq p \leq n$, where $d\sigma$ denotes the rotationally invariant probability measure on the unit sphere.

It is now known, thanks to an observation of Bourgain [1], that any progress towards the resolution of the analytic formulation automatically implies progress towards the geometric formulation. In particular, if the Kakeya maximal function is bounded on L^p , it follows that Kakeya sets have Hausdorff (and, hence, upper and lower Minkowski) dimension at least p .

Modern investigations of this conjecture essentially date back to the 1970's, when Davies [5] proved that any Kakeya set in \mathbb{R}^2 has Hausdorff dimension (and, hence, upper and lower Minkowski dimension) 2. Later, Córdoba [4] proved the analogous result for the maximal function in \mathbb{R}^2 . Though these two results essentially resolved all questions concerning the Kakeya conjecture in 2 dimensions, the conjecture remains open in dimensions 3 and higher.

The (n, k) problem is a variant of the Kakeya problem in which one replaces lines with k -planes. In order to formally describe the (n, k) problem, we need a few definitions.

Definition 1.5. A set $E \subset \mathbb{R}^n$ is said to be an (n, k) set if for any k -dimensional subspace $\pi \subset \mathbb{R}^n$, there exists a k -dimensional unit cube Q_π parallel to π such that $Q_\pi \subset E$.

Clearly, an $(n, 1)$ set is simply a Kakeya set. The geometric conjecture associated with these sets is essentially the same as the Kakeya conjecture:

Conjecture 1.6. *If $E \subset \mathbb{R}^n$ is an (n, k) set, then $\dim(E) = n$.*

Once again, dimension can be taken as upper Minkowski, lower Minkowski or Hausdorff. One can also define the (n, k) maximal function:

Definition 1.7. Given positive integers $1 \leq k < n$, let $G(n, k)$ denote the Grassmannian manifold of all k -dimensional subspaces of \mathbb{R}^n . For a function $f \in L^1_{loc}(\mathbb{R}^n)$ and a real number $0 < \delta \ll 1$, we define the (n, k) maximal function as

$$T_{n,k,\delta}f : G(n, k) \rightarrow \mathbb{R}$$

$$T_{n,k,\delta}f(\pi) = \sup_{a \in \mathbb{R}^n} \frac{1}{|P_\delta(a, \pi)|} \int_{P_\delta(a, \pi)} |f(x)| dx$$

where $P_\delta(a, \pi)$ denotes the δ -neighborhood of the k -dimensional unit cube parallel to π and centered at $a \in \mathbb{R}^n$.

Using this definition, the analytic version of the (n, k) conjecture is then

Conjecture 1.8. *For all $\epsilon > 0$, and $0 < \delta \ll 1$, one has*

$$\|T_{n,k,\delta}f\|_{L^p(G(n,k), d\pi)} \leq C_\epsilon \delta^{\frac{n}{p} - k - \epsilon} \|f\|_{L^p(\mathbb{R}^n, dx)} \quad (2)$$

for $1 \leq p \leq \frac{n}{k}$, where $d\pi$ denotes the rotationally invariant probability measure on $G(n, k)$.

Though a large amount of work has been devoted to the (n, k) problems mentioned here, as well as various other k -plane transforms (for example, see the work of Strichartz [21], Christ [3] and Drury [6]), the previous results most relevant to this paper are those due to Oberlin and Stein [18], Themis Mitsis [15],[16], as well as Wolff's bound for the Kakeya maximal function [22]

We investigate the problem in the setting of vector spaces over finite fields; this slightly unconventional choice of vector spaces was inspired by a recent paper

due to Mockenhaupt and Tao [17]. When working in a discrete setting, one is allowed to focus almost exclusively on the geometric and combinatorial aspects of the problem. In particular, problems which arise in the study of Kakeya-type problems in Euclidean spaces due to multiplicity of scales (in particular, the troublesome behavior of δ -tubes which intersect at small angles) are essentially non-existent in the discrete setting. Furthermore, working in this setting allows one to import useful tools from other areas of mathematics, such as analytic number theory (Gauss sums, Kloosterman sums) and incidence geometry. There are, of course, a few negative side-effects resulting from the discretization of the problem. For example, Taylor approximations of surfaces don't make sense in F^n , and there is no chance to make use of arguments requiring induction on scales.

Before stating the main result, a bit of notation is necessary. Given a finite field F , we let $G(n, k)$ denote the (Grassmannian) set of all k -dimensional subspaces of the vector space F^n . We will always assume that F is a very large finite field. Aside from the fact that the results discussed here are somewhat trivial when F is small, working with large fields allows one to develop an intuition regarding the Minkowski dimension of the geometric objects we will be studying. When considering Lebesgue spaces of functions defined on $G(n, k)$, we endow this set with a normalized counting measure $d\nu$, so that for any subset $\Pi \subset G(n, k)$, $\nu(\Pi) = |F|^{-k(n-k)}|\Pi|$ (where $|\cdot|$ simply denotes cardinality). The vector space F^n is endowed with a standard counting measure dx . With this notation, we may define the (n, k) maximal function operator as

$$T_{n,k}f(\pi) = \sup_{x \in F^n} \sum_{y \in x + \pi} |f(y)| \quad (3)$$

where f is some real-valued function on F^n , and $\pi \in G(n, k)$.

Our primary objective here will be to determine a class of Lebesgue spaces on which this operator is “bounded”. Of course, since we are working in vector spaces over finite fields, we won't be encountering any divergent integrals. Our definition of boundedness will be as follows:

Definition 1.9. Let $T_{n,k}(p \rightarrow q)$ denote the smallest quantity such that

$$\|T_{n,k}f\|_{L^q(d\nu)} \leq T_{n,k}(p \rightarrow q)\|f\|_{L^p(dx)}$$

holds for all functions $f : F^n \rightarrow \mathbb{R}$. We say that $T_{n,k}$ is *bounded* from L^p to L^q if $T_{n,k}(p \rightarrow q) \lesssim 1$ as $|F| \rightarrow \infty$.

The notation $A \lesssim B$ in the above definition means that for all $\epsilon > 0$, there exists some constant C_ϵ such that $A \leq C_\epsilon |F|^\epsilon B$. Similarly, $A \lesssim B$ indicates $A \leq CB$ for some constant C independent of the field F . Also, we say $A \approx B$ (respectively $A \sim B$) if $A \lesssim B$ and $B \lesssim A$ (respectively $A \lesssim B$ and $B \lesssim A$).

One can immediately observe a few necessary conditions on p and q by examining certain counterexamples. For example, observing the function $f \equiv 1$ shows that one must have $p \leq \frac{n}{k}$. Also, considering the function $f = \chi_\pi$ for some k -plane $\pi \subset F^n$ will reveal that one must have $q \leq (n-k)p'$. It is conjectured that these necessary conditions are also sufficient, though only partial results are currently available.

Our main result will be a proof of the following result for the (n, k) maximal function by means of incidence combinatorial techniques:

Theorem 1.10. *Let n, k be positive integers such that $2 \leq k \leq n-2$. The operator $T_{n,k} : L^p(F^n) \rightarrow L^q(G(n, k))$ is bounded when $1 \leq p \leq \frac{kn+k+1}{k(k+1)}$ and $1 \leq q \leq (n-k)p'$.*

As an immediate corollary, we have the following result concerning the geometric version of the (n, k) problem:

Corollary 1.11. *Let $E \subset F^n$ be an (n, k) set, where $2 \leq k \leq n-2$ are integers. Then,*

$$|E| \gtrsim |F|^{\frac{kn+k+1}{k+1}}. \quad (4)$$

Remark 1.12. It should be noted that Theorem 1.10 is in no way the best known result for the (n, k) problem. In the Euclidean case, Bourgain [1] showed that (n, k) sets have positive n -dimensional Lebesgue measure whenever $n \leq 2^{k-1} + 2$. Furthermore, Oberlin ([19], [20]) has generalized this result to the corresponding maximal function estimate, and improved the result for certain values of n and k by making use of recent advances concerning the X-ray transform ([23], [13], [10]). The novelty of Theorem 1.10 is in its proof; the proof presented here is entirely combinatorial, whereas the work of Bourgain and Oberlin incorporates tools from Fourier analysis, thus leading to stronger results.

This result might be interpreted as an extension of the work of Wolff [22], who showed that $(n, 1)$ sets have cardinality $\gtrsim |F|^{\frac{n+2}{2}}$, and Oberlin and Stein [18], whose work implies that $(n, n-1)$ sets have cardinality $\sim |F|^n$. A Euclidean version of the case $k = 2$ was first proven by Mitsis [15]. His proof is an adaptation of the “hairbrush” construction used by Wolff to demonstrate a bound for the Kakeya maximal function. For our purposes, we shall be following the example set by Mockenhaupt and Tao [17] who find an alternate proof of Wolff’s Kakeya result by incidence combinatorial methods¹.

The paper will be organized as follows: In section 2, we will introduce the main combinatorial tools which will be used in later computations, and establish a correspondence between bounds on the number of incidences between points and

¹It is, of course, possible to adapt Wolff’s method to provide the same bound stated in Theorem 1.10. Furthermore, the two methods are, in a certain sense, “equivalent”. For more details, the reader is directed to [2].

k -planes and estimates for the (n, k) maximal function as defined in equation (3). Next, in section 3, we will prove a generalized version of Wolff's "two-ends reduction". This reduction will allow us to prove non-trivial incidence bounds without being hindered by the existence of certain pathological configurations of points and k -planes. The remaining sections will be devoted to proving an incidence bound between points and k -planes. We shall arrive at this bound by estimating the number of $(k+1)$ -simplices arising from a configuration of points P and k -planes Π which have vertices from P and faces from Π . The lower bound for the number of such simplices is computed by an inductive procedure, and is addressed in sections 4 and 5. The upper bound for the number of simplices is computed in section 6.

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2 Preliminary Incidence Combinatorial Techniques

As the proof of the main theorem in this paper will be largely combinatorial, our first task will be to establish some machinery designed to translate incidence combinatorial results into maximal function estimates, and vice versa.

An important combinatorial tool which will be used extensively the following set-theoretical version of Cauchy-Schwarz:

Theorem 2.1 (Cauchy-Schwarz). *Let A and B be finite sets with some relation $a \sim b$ between elements $a \in A$ and $b \in B$. Then,*

$$\left| \left\{ (a, a', b) \in A \times A \times B : \begin{array}{l} a \sim b \\ a' \sim b \end{array} \right\} \right| \geq \frac{|\{(a, b) \in A \times B : a \sim b\}|^2}{|B|}. \quad (5)$$

Proof. To see how Theorem 2.1 follows from the traditional Cauchy-Schwarz inequality, define a function f on the set B as

$$\begin{aligned} f(b) &= |\{a \in A : a \sim b\}| \\ &= \sum_{a \in A} \chi_{a \sim b}. \end{aligned}$$

Next, using Cauchy-Schwarz, we have

$$\left(\sum_B f(b) \right)^2 \leq |B| \|f\|_{L^2(B)}^2,$$

where the L^2 norm is computed with respect to counting measure on the finite set B . Theorem 2.1 then follows by observing that

$$\left(\sum_B f(b)\right)^2 = |\{(a, b) \in A \times B : a \sim b\}|^2,$$

and

$$\|f\|_{L^2(B)}^2 = |\{(a, a', b) \in A \times A \times B : a \sim b, a' \sim b\}|.$$

□

In practice, the sets A and B will always denote certain configurations of points, planes and lines, and the relation \sim will denote some sort of geometric incidence. For example, given a configuration of points P and k -planes Π , we may obtain a lower bound on the number of “double point-plane incidences” in terms of the numbers of point-plane incidences and planes:

$$|\{(p, p', \pi) \in P \times P \times \Pi : p, p' \in \pi\}| \geq \frac{|\{(p, \pi) \in P \times \Pi : p \in \pi\}|^2}{|\Pi|}. \quad (6)$$

Of course, this idea can be generalized into a version of Hölder’s inequality:

$$|\{(p_1, \dots, p_m, \pi) \in P^m \times \Pi : p_i \in \pi\}| \geq \frac{|\{(p, \pi) \in P \times \Pi : p \in \pi\}|^m}{|\Pi|^{m-1}}. \quad (7)$$

The set appearing on the right hand side of both (6) and (7) is known as the *incidence set* associated to the points P and k -planes Π . As this set will be appearing quite often throughout this paper, we’ll introduce the following notation:

$$I(P, \Pi) := \{(p, \pi) \in P \times \Pi : p \in \pi\} \quad (8)$$

Often, the arguments P and Π will be suppressed when they are obvious from the context.

The following correlation between maximal function estimates and incidence bounds is used in [17] to demonstrate a bound for the Keakey maximal function, and we generalize it to suit our purposes as follows:

Proposition 2.2. *Given exponents $1 \leq p, q \leq \infty$, the bound $T_{n,k}(p \rightarrow q) \lesssim 1$ holds if and only if given any collection of points $P \subset F^n$ and any direction separated collection of k -planes Π contained in F^n , the following incidence bound holds:*

$$|\{(p, \pi) \in P \times \Pi : p \in \pi\}| \lesssim |P|^{\frac{1}{p}} |\Pi|^{\frac{1}{q'}} |F|^{\frac{k(n-k)}{q}}. \quad (9)$$

Before proceeding with the proof, note that the conjectured best possible incidence bound (corresponding to the necessary conditions $p \leq \frac{n}{k}$ and $q \leq (n-k)p'$) is

$$|I(P, \Pi)| \lesssim |P|^{\frac{k}{n}} |\Pi|^{\frac{n-1}{n}} |F|^{\frac{k(n-k)}{n}}. \quad (10)$$

This expression will appear several times throughout the course of the paper.

Proof. First, we assume $T_{n,k}(p \rightarrow q) \lesssim 1$. Let P and Π be as in the statement of the proposition, and let $D \subset G(n, k)$ denote the direction set of Π (i.e. each $\pi \in \Pi$ is a parallel translate of exactly one element of D). Then, the result simply follows from Hölder's inequality:

$$\begin{aligned} |\{(p, \pi) \in P \times \Pi: p \in \pi\}| &\leq \sum_{d \in D} T_{n,k} \chi_P(\pi(d)) \\ &= |F|^{k(n-k)} \int_D T_{n,k} \chi_P d\nu \\ &\lesssim |F|^{k(n-k)} \left(\frac{|\Pi|}{|F|^{k(n-k)}} \right)^{\frac{1}{q'}} |P|^{\frac{1}{p}}. \end{aligned}$$

To prove the converse, it suffices (by duality) to show

$$\sum_{x \in F^n} \int_{G(n,k)} g(\sigma) \chi_{\sigma+x_0(\sigma)}(x) f(x) d\nu(\sigma) \lesssim \|g\|_{L^{q'}(d\sigma)} \|f\|_{L^p(dx)}$$

for any pair of functions f and g (defined on F^n and $G(n, k)$, respectively), where x_0 is some function which translates elements of $G(n, k)$ to affine position. Since our notation allows us to lose factors of $\log|F|$, we may employ the dyadic pigeonhole principle, and assume that $f = \chi_P$ and $g = \chi_D$ for sets $P \subset F^n$, and $D \subset G(n, k)$. After making these simplifications, we must now show

$$\frac{1}{|F|^{k(n-k)}} \sum_{x \in P} \sum_{\sigma \in D} \chi_{\sigma+x_0(\sigma)}(x) \lesssim \left(\frac{|D|}{|F|^{k(n-k)}} \right)^{\frac{1}{q'}} |P|^{\frac{1}{p}}.$$

Now, simply define the collection of k -planes to be $\Pi := \{\sigma + x_0 : \sigma \in D\}$. The above equation then follows directly from (9). \square

3 Avoiding Obstructions To Nontrivial Incidence Bounds

In this section, we address an issue which often causes problems when counting incidences between points and (generally speaking) algebraic varieties of dimension greater than 1: it is quite easy to construct a large set of points P and a large set of planes Π such that every point is contained in every plane. The classic example of this behavior can be seen in the following example:

Definition 3.1. A *type-(2, 1) degenerate configuration* is a configuration of a set of points P and a set of 2-planes Π such that all of the points in P lie on some line l , and all of the planes in Π contain the line l .

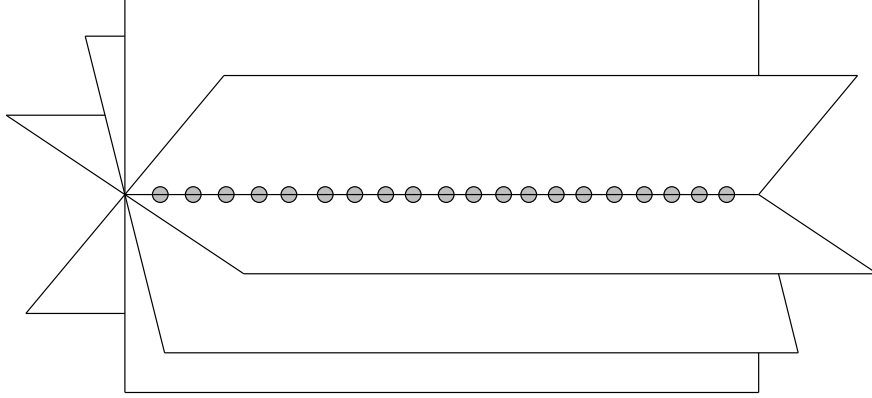


Figure 1: A type-(2, 1) degenerate configuration.

This configuration is notable because the incidence set associated to it actually attains the worst possible upper bound, $|I| = |\Pi||P|$. So, it seems that this counterexample should prohibit us from obtaining any sort of nontrivial upper bound on the size of the incidence set which would hold for all possible collections of points and planes. For higher values of k , there are even more possible degenerate configurations of this type; given a positive integer $r < k$, one can consider a family of $|F|^r$ points which all lie on some affine r -plane σ , and a direction separated collection $|F|^{(k-r)(n-k)}$ k -planes which all contain σ . We shall call such a configuration *type-(k, r) degenerate*.

Because of the existence of such configurations, it is necessary to place some (presumably mild) restrictions on distributions of points and/or planes in question. In recent work concerning incidences between points and surfaces, (for example, L     and Solymosi [12], and Elekes and T     [8]), point sets were assumed to have certain uniformity properties to prohibit pathological configurations. For our purposes, however, we shall work with arbitrary distributions of points in F^n , and exploit the fact that any family of planes in consideration must be direction separated.

The following proposition acts in the same way as the “two-ends reduction” first used by Wolff [22] to improve estimates for the Kakeya maximal function. We will eliminate the threat posed by configurations in which points tend to cluster along low dimensional subsets of k -planes from Π by eliminating the possibility of having sparsely populated k -planes. The actual proof of this new version of the reduction for the (n, k) problem is quite different from Wolff’s; it is more

closely related to Drury's work on the X-ray transform [7].

Proposition 3.2 (Generalized Two-ends Reduction). *Let $P \subset F^n$ be a collection of points, and Π a direction separated collection k -planes in F^n such that*

$$|I(P, \Pi)| \lesssim |\Pi| |F|^{k-1}, \quad (11)$$

and

$$|P \cap \pi| \approx \frac{|I(P, \Pi)|}{|\Pi|} \text{ for each } \pi \in \Pi. \quad (12)$$

Then, the best possible incidence bound holds:

$$|I(P, \Pi)| \lesssim |P|^{\frac{k}{n}} |\Pi|^{\frac{n-1}{n}} |F|^{\frac{k(n-k)}{n}}. \quad (13)$$

Proof. The proof of this proposition will be inductive. We shall first show that (13) holds under the assumption $|I| \lesssim |\Pi|$ (hence it is safe to assume $|I| \gg |\Pi|$), and then show that the result holds when $|I| \lesssim |\Pi| |F|^r$ for any integer $0 \leq r \leq k-1$.

Suppose $|I| \lesssim |\Pi|$. Then, making use of the trivial estimates $|\Pi| \lesssim |F|^{k(n-k)}$ and $|P| \geq 1$, we have

$$\begin{aligned} |I| &\lesssim |\Pi|^{\frac{n-1}{n}} |F|^{\frac{k(n-k)}{n}} \\ &\leq |P|^{\frac{k}{n}} |\Pi|^{\frac{n-1}{n}} |F|^{\frac{k(n-k)}{n}}. \end{aligned}$$

Therefore, we may assume $|I| \gg |\Pi|$.

Next, assume that $|\Pi| |F|^{r-1} \ll |I| \lesssim |\Pi| |F|^r$ for some positive integer $r \leq k-1$. In order to arrive at the desired conclusion, we will estimate the size of the following set

$$J_r \equiv J_r(P, \Pi) := \{(p_0, \dots, p_r, \pi) \in P^{r+1} \times \Pi : p_i \in \pi \text{ for each } i\}. \quad (14)$$

Using Hölder's inequality, we arrive at a lower bound of

$$|J_r| \geq \frac{|I|^{r+1}}{|\Pi|^r}. \quad (15)$$

In order to compute a corresponding upper bound, we break the set J_r into a union of disjoint subsets in the following manner:

$$J_r = \bigcup_{j=0}^r J_r^{(j)}$$

where

$$J_r^{(j)} := \{(p_0, \dots, p_r, \pi) \in J_r : \dim[p_0, \dots, p_r] = j\}$$

Now we estimate each of these subsets separately. Clearly, we have

$$|J_r^{(0)}| = |I|. \quad (16)$$

Next, when $1 \leq j \leq r-1$, we have $|\Pi|$ choices for the k -plane, $\approx \left(\frac{|I|}{|\Pi|}\right)^{j+1}$ choices for a $(j+1)$ -tuple of points which spans $[p_0, \dots, p_r]$, and $|F|^{j(r-j)}$ choices for the remaining points. Therefore, we have

$$|J_r^{(j)}| \lesssim \frac{|I|^{j+1}}{|\Pi|^j} |F|^{j(r-j)}. \quad (17)$$

Finally, in order to estimate $|J_r^{(r)}|$, we observe that there are $\sim |P|^{r+1}$ choices for the $(r+1)$ -tuple of points, and (because the collection of k -planes is direction separated) there are at most $\sim |F|^{(k-r)(n-k)}$ k -planes for the collection Π which can contain the r -dimensional affine space spanned by the already chosen $(r+1)$ -tuple of points. Putting together all of these observations, we have

$$\frac{|I|^{r+1}}{|\Pi|^r} \leq |J_r| \lesssim |P|^{r+1} |F|^{(k-r)(n-k)} + |I| + \sum_{j=1}^{r-1} \frac{|I|^{j+1}}{|\Pi|^j} |F|^{j(r-j)}. \quad (18)$$

Furthermore, since we are assuming $|I| \gg |\Pi| |F|^{r-1}$, we have

$$\frac{|I|^{r+1}}{|\Pi|^r} \gg \frac{|I|^{j+1}}{|\Pi|^j} |F|^{j(r-j)} \quad (19)$$

whenever $1 \leq j \leq r-1$. Similarly, $\frac{|I|^{r+1}}{|\Pi|^r} \gg |I|$. Therefore, the first term on the right hand side of equation (18) is dominant, and we have

$$\frac{|I|^{r+1}}{|\Pi|^r} \lesssim |P|^{r+1} |F|^{(k-r)(n-k)}. \quad (20)$$

A bit of algebraic manipulation then leads us to the following incidence bound:

$$|I| \lesssim |P| |\Pi|^{\frac{r}{r+1}} |F|^{\frac{(k-r)(n-k)}{r+1}}. \quad (21)$$

Finally, we do a bit more algebra, and observe that

$$\begin{aligned} |I| &\lesssim \min\{|P| |\Pi|^{\frac{r}{r+1}} |F|^{\frac{(k-r)(n-k)}{r+1}}, |\Pi| |F|^r, |P| |\Pi|\} \\ &\lesssim |P|^{\frac{k}{n}} |\Pi|^{\frac{n-1}{n}} |F|^{\frac{k(n-k)}{n}}. \end{aligned}$$

Since this is true whenever $1 \leq r \leq k-1$, we are done. \square

Remark 3.3. It should be noted that in the special case when $|P| = |F|^r$ and $|\Pi| = |F|^{(k-r)(n-k)}$, all four of the quantities $|P||\Pi|$, $|\Pi||F|^r$, $|P||\Pi|^{\frac{r}{r+1}}|F|^{\frac{(k-r)(n-k)}{r+1}}$ and $|P|^{\frac{k}{n}}|\Pi|^{\frac{n-1}{n}}|F|^{\frac{k(n-k)}{n}}$ coincide. These values for $|P|$ and $|\Pi|$ are, of course, the same values that one would see in the construction of a type- (k, r) degenerate configuration, as was described earlier in this section. These configurations, therefore, are examples of situations in which the worst possible incidence bound coincides with the best possible incidence bound.

Now that we have eliminated the potential problems caused by the existence of sparsely populated k -planes, we are in a position to formulate a more flexible version of Proposition 2.2. In order to make use of Proposition 3.2 in what follows, we shall need to establish some notation. Given a collection of points P and a direction separated collection of k -planes Π , it is clear that the following equality holds:

$$|I| = \sum_{\pi \in \Pi} |P \cap \pi|.$$

Therefore, by pigeonholing, there exists a subcollection $\tilde{\Pi} \subset \Pi$ such that

$$|P \cap \pi| \approx \frac{|I|}{|\tilde{\Pi}|} \text{ for each } \pi \in \tilde{\Pi}. \quad (22)$$

The only control we have over the size of the set $\tilde{\Pi}$ is the trivial estimate $|\tilde{\Pi}| \leq |\Pi|$. Furthermore, letting \tilde{I} denote the incidence set $I(P, \tilde{\Pi})$, we have

$$|\tilde{I}| = \sum_{\pi \in \tilde{\Pi}} |P \cap \pi| \approx |I|. \quad (23)$$

With this notation in place, we may now state and prove the following proposition:

Proposition 3.4. *Let $0 \leq a, b, c \leq 1$ be real numbers such that $(n-k)b + c \geq 1$. If the following incidence bound holds*

$$|\tilde{I}| \lesssim |P|^a |\tilde{\Pi}|^{1-b} |F|^{k(1-c)} + |P||\tilde{\Pi}|^{\frac{k-1}{k}} + |\tilde{\Pi}||F|^{k-1}, \quad (24)$$

then the statement $T_{n,k}(p \rightarrow q) \lesssim 1$ is true with $p = \frac{(n-k)b+c}{a}$ and $q = \min \left\{ (n-k)p', \frac{(n-k)b+c}{b} \right\}$.

Remark 3.5. Unfortunately, the condition $(n-k)b + c \geq 1$ holds only when $n \geq k+2$ with the values of a , b and c to be computed in the next section. This technicality prevents us from using the methods described here to demonstrate an entirely geometric proof of the result of Oberlin and Stein.

Proof. To start with, if the term $|P||\tilde{\Pi}|^{\frac{k-1}{k}}$ dominates, then we arrive at the best possible bound:

$$|\tilde{I}| \lesssim (|P||\tilde{\Pi}|^{\frac{k-1}{k}})^{\frac{k}{n}} (|\tilde{\Pi}||F|^k)^{\frac{n-k}{n}} = |P|^{\frac{k}{n}} |\tilde{\Pi}|^{\frac{n-1}{n}} |F|^{\frac{k(n-k)}{n}}.$$

So, we may also assume $|\tilde{I}| \gg |P||\tilde{\Pi}|^{\frac{k-1}{k}}$.

If the term $|\tilde{\Pi}||F|^{k-1}$ dominates the right-hand side of equation (24), then (since our refinements allow us to assume equation (12) holds) Proposition 3.2 shows that we obtain the best possible incidence bound. Therefore, we may assume $|\tilde{I}| \gg |\tilde{\Pi}||F|^{k-1}$. To finish the proof, assume $|\tilde{I}| \lesssim |P|^a |\tilde{\Pi}|^{1-b} |F|^{k(1-c)}$. Since we have refined our sets in such a way that $|\tilde{I}| \approx |I|$ and $|\tilde{\Pi}| \leq |\Pi|$, we have $|I| \lesssim |P|^a |\Pi|^{1-b} |F|^{k(1-c)}$. Taking a convex combination (making use of the assumption $(n-k)b + c \geq 1$) of this estimate with the trivial estimate $|I| \leq |\Pi||F|^k$, and applying Proposition 2.2 completes the proof. \square

4 Simplex Construction Part One: The Lower Bound

The object of the next three sections will be to prove the following incidence bound:

Theorem 4.1. *Let $P \subset F^n$ be a collection of points and Π a direction separated collection of k -planes contained in F^n . Then,*

$$|\tilde{I}| \lesssim |P|^{\frac{k(k+1)}{k^2+2k+2}} |\tilde{\Pi}|^{\frac{k^2+k+2}{k^2+2k+2}} |F|^{\frac{k(k+1)}{k^2+2k+2}} + |P||\tilde{\Pi}|^{\frac{k-1}{k}} + |\tilde{\Pi}||F|^{k-1} \quad (25)$$

where the sets \tilde{I} and $\tilde{\Pi}$ are as described in equations (22) and (23).

Once this incidence bound has been demonstrated, we may apply Proposition 3.4 to obtain the desired bound for the (n, k) maximal function. The proof presented here is inspired by an argument found in [17] to prove a similar incidence bound in the case $k = 1$; for their result, they obtain upper and lower bounds on the number of triangles appearing in the configuration of points and lines (with sides from the given collection of lines, and vertices from the given collection of points). For our purposes here, we will be obtaining upper and lower bounds on the number of $(k+1)$ -simplices appearing in the configuration, each with $k+2$ k -dimensional faces from the collection $\tilde{\Pi}$, and $k+2$ vertices coming from the collection of points P . The proof of the lower bound will be an induction on k (making use of the fact that a $(k+1)$ -simplex is the cone of k -simplex). The upper bound will make use of the direction separatedness of the family of k -planes.

We begin by demonstrating a lower bound for the number of $(k+1)$ -simplices arising from collections of points P and k -planes $\tilde{\Pi}$ satisfying certain hypotheses. The actual statement of this lower bound will require quite a bit of terminology and notation, so we begin with a few definitions.

Definition 4.2. Let n and k be integers such that $1 \leq k \leq n-2$. Given a collection of points $P \subset F^n$ and k -planes $\tilde{\Pi}$, we say that this configuration of points and planes satisfies hypothesis $H1(k, P, \tilde{\Pi})$ if

$$|\tilde{I}| \gg |P||\tilde{\Pi}|^{\frac{k-1}{k}}. \quad (26)$$

Definition 4.3. Let n and k be integers such that $1 \leq k \leq n-2$. Given a collection of points $P \subset F^n$ and k -planes $\tilde{\Pi}$, we say that this configuration of points and planes satisfies hypothesis $H2(k, P, \tilde{\Pi})$ if

$$|\tilde{I}| \gg |\tilde{\Pi}| |F|^{k-1}. \quad (27)$$

Observe that these hypotheses are derived from the error terms from the main incidence bound we aim to demonstrate (25). As the computation of the lower bound the collection of simplices will be inductive, we shall need to investigate the behavior of these hypotheses for varying values of k . Also, observe that we have not yet mentioned the hypothesis that the set $\tilde{\Pi}$ be direction separated; this hypothesis will not be used until section 6 when computing the upper bound on the collection of simplices, and its absence from the lower bound computation greatly simplifies the induction process.

In order to carry out the induction, we shall need to construct a few sets from our base sets of P , $\tilde{\Pi}$ and \tilde{I} . The first step towards carrying out these constructions will be observing that the assumptions $|P \cap \pi| \approx \frac{|I|}{|\tilde{\Pi}|}$ for each $\pi \in \tilde{\Pi}$ and $|\tilde{I}| \approx |I|$ permit us to make use of Proposition 3.2. This allows us to assume that $|\tilde{I}| \gg |\tilde{\Pi}| |F|^{k-1}$. Furthermore, we may use the simple argument found in the proof of Proposition 3.4 to assume that $|\tilde{I}| \gg |P||\tilde{\Pi}|^{\frac{k-1}{k}}$. In other words, the statements $H1(k, P, \tilde{\Pi})$ and $H2(k, P, \tilde{\Pi})$ are true.

Next, we construct the following set:

$$\tilde{I}'_k := \{(\pi, p_1, \dots, p_k) \in \tilde{\Pi} \times P^k : p_i \in \pi \text{ for all } i, \dim[p_1, \dots, p_k] = k-1\} \quad (28)$$

Observe that Hölder's inequality, along with Proposition 3.2, gives us a lower bound on $|\tilde{I}'_k|$ of

$$|\tilde{I}'_k| \gtrsim \frac{|\tilde{I}|^k}{|\tilde{\Pi}|^{k-1}} \quad (29)$$

This set also needs to be refined a bit; we wish to remove elements of the set \tilde{I}'_k which are degenerate in the sense that the $(k-1)$ -plane spanned by the

k -tuple (p_1, \dots, p_k) carries a small number of points. To make this refinement, we introduce a relation \sim on the set $\tilde{I}'_k \times P$ defined as

$$(\pi, p_1, \dots, p_k) \sim q \text{ if } q \in [p_1, \dots, p_k]. \quad (30)$$

Since each $\pi \in \tilde{\Pi}$ contains $\approx \frac{|\tilde{I}|}{|\tilde{\Pi}|}$ points from P , one should expect each $(k-1)$ -dimensional slice of π to contain roughly $\frac{|\tilde{I}|}{|\tilde{\Pi}||F|}$ points from P . With this intuition, we may refine \tilde{I}'_k as follows:

$$\tilde{I}_k := \{i_k \in \tilde{I}'_k : |\{q \in P : i_k \sim q\}| \geq \frac{|\tilde{I}|}{10|\tilde{\Pi}||F|}\} \quad (31)$$

To see that this refinement is okay, we need only show that the set $\tilde{I}'_k \setminus \tilde{I}_k$ is small. A simple estimate will take care of this: we have $|\tilde{\Pi}|$ choices for the k -plane, each of these k -planes contains $\sim |F|^k$ hyperplanes, and each element of the set $\tilde{I}'_k \setminus \tilde{I}_k$ can have at most $\left(\frac{|\tilde{I}|}{10|\tilde{\Pi}||F|}\right)^k$ k -tuples of points on any of these hyperplanes. So:

$$|\tilde{I}'_k \setminus \tilde{I}_k| \lesssim \left(\frac{|\tilde{I}|}{10|\tilde{\Pi}||F|}\right)^k |F|^k |\tilde{\Pi}| = \frac{|\tilde{I}|^k}{10^k |\tilde{\Pi}|^{k-1}} \ll |\tilde{I}'_k|.$$

This allows us to replace \tilde{I}'_k with \tilde{I}_k without doing much harm.

Next, we will take pairs of elements of the set \tilde{I}_k , and identify them along their k -tuples:

$$V_k' := \{((\pi_0, p_1, \dots, p_k), (\pi, q_1, \dots, q_k)) \in \tilde{I}_k \times \tilde{I}_k : p_i = q_i \text{ for all } i\}. \quad (32)$$

An application of Cauchy-Schwarz gives us a lower bound on the size of this set:

$$|V_k'| \geq \frac{|\tilde{I}_k|^2}{|P|^k} \quad (33)$$

This set also needs to be refined; we wish to remove elements from V_k' which are degenerate in the sense that $\pi_0 = \pi$. Such an element of V_k' is, in fact, an element of \tilde{I}_k , so this refinement will be okay if we can show that $|V_k'| \gg |\tilde{I}_k|$. In order to show this, we make use of hypothesis H1($k, P, \tilde{\Pi}$):

$$|V_k'| \geq \frac{|\tilde{I}_k|^2}{|P|^k} \gtrsim |\tilde{I}_k| \frac{|\tilde{I}|^k}{|P|^k |\tilde{\Pi}|^{k-1}} \gg |\tilde{I}_k|.$$

So, if we define the set V_k as

$$V_k := \{(\pi_0, \pi, p_1, \dots, p_k) \in V_k' : \pi_0 \neq \pi\}, \quad (34)$$

then we have $|V_k| \gtrsim |V_k'| \gtrsim \frac{|\tilde{I}_k|^2}{|P|^k}$.

Next, to each element of V_k we wish to add a point from P which lives in $\pi \setminus [p_1, \dots, p_k]$.

$$V_{k,p} := \{((\pi_0, \pi, p_1, \dots, p_k), x) \in V_k \times P : x \in P \cap (\pi \setminus [p_1, \dots, p_k])\} \quad (35)$$

Since there are $\approx \frac{|\tilde{I}|}{|\tilde{\Pi}|}$ points on each plane, and they do not cluster along low dimensional subspaces by Proposition 3.2, we have the following lower bound

$$|V_{k,p}| \gtrsim |V_k| \frac{|\tilde{I}|}{|\tilde{\Pi}|}. \quad (36)$$

One last refinement is needed before we can proceed with the construction of simplices. For a given pair $(\pi_0, x) \in \tilde{\Pi} \times P$ such that $x \notin \pi_0$, we define a function $f(\pi_0, x)$ as

$$f(\pi_0, x) := |\{(\pi, p_1, \dots, p_k) \in \tilde{I}_k : ((\pi_0, \pi, p_1, \dots, p_k), x) \in V_{k,p}\}|.$$

Then,

$$\sum f(\pi_0, x) = |V_{k,p}| \gtrsim |V_k| \frac{|\tilde{I}|}{|\tilde{\Pi}|},$$

so we may pigeonhole this sum to find a family of plane-point pairs (π_0, x)

$$D(P, \tilde{\Pi}) := \{(\pi_0, x) \in \tilde{\Pi} \times P : x \notin \pi_0, f(\pi_0, x) \gtrsim |V_k| \frac{|\tilde{I}|}{|\tilde{\Pi}| |D(P, \tilde{\Pi})|}\}. \quad (37)$$

As was the case for the parameter $|\tilde{\Pi}|$, we have no control over the parameter $|D(P, \tilde{\Pi})|$ other than the trivial upper bound $|D(P, \tilde{\Pi})| \leq |P| |\tilde{\Pi}|$.

Remark 4.4. It is somewhat disconcerting that the value of the function $f(\pi_0, x)$ may be very small on our set $D(P, \tilde{\Pi})$. This potential problem, however, will be ruled out in Section 5.

In summary, we have made the following constructions:

$$\begin{aligned}
\tilde{I}_k &= \left\{ (\pi, p_1, \dots, p_k) \in \tilde{\Pi} \times P^k : \begin{array}{l} p_i \in \pi \text{ for each } i, \\ \dim[p_1, \dots, p_k] = k-1 \\ |P \cap [p_1, \dots, p_k]| \geq \frac{|\tilde{I}|}{10|\tilde{\Pi}||F|} \end{array} \right\} \\
|\tilde{I}_k| &\gtrsim \frac{|\tilde{I}|^k}{|\tilde{\Pi}|^{k-1}} \\
V_k &= \left\{ ((\pi_0, p_1, \dots, p_k), (\pi, q_1, \dots, q_k)) \in \tilde{I}_k \times \tilde{I}_k : \begin{array}{l} p_i = q_i \text{ for each } i \\ \pi \neq \pi_0 \end{array} \right\} \\
|V_k| &\gtrsim \frac{|\tilde{I}_k|^2}{|P|^k} \\
V_{k,p} &= \{((\pi_0, \pi, p_1, \dots, p_k), x) \in V_k \times P : x \in P \cap (\pi \setminus [p_1, \dots, p_k])\} \\
|V_{k,p}| &\gtrsim |V_k| \frac{|\tilde{I}|}{|\tilde{\Pi}|} \\
f(\pi_0, x) &= |\{(\pi, p_1, \dots, p_k) \in \tilde{I}_k : ((\pi_0, \pi, p_1, \dots, p_k), x) \in V_{k,p}\}| \\
\sum f(\pi_0, x) &= |V_{k,p}| \gtrsim |V_k| \frac{|\tilde{I}|}{|\tilde{\Pi}|} \\
D(P, \tilde{\Pi}) &= \{(\pi_0, x) \in \tilde{\Pi} \times P : x \notin \pi_0, f(\pi_0, x) \gtrsim |V_k| \frac{|\tilde{I}|}{|\tilde{\Pi}||D(P, \tilde{\Pi})|}\} \\
|D(P, \tilde{\Pi})| &\leq |P||\tilde{\Pi}|
\end{aligned}$$

With this notation established, we may finally begin to compute a lower bound on the number of simplices.

Lemma 4.5. *Given an arrangement of points P and k -planes $\tilde{\Pi}$ satisfying hypotheses $H1(k, P, \tilde{\Pi})$ and $H2(k, P, \tilde{\Pi})$, let $S_k(P, \tilde{\Pi})$ denote the set of $(k+1)$ -simplices in F^n with faces from $\tilde{\Pi}$ and vertices from P . Then,*

$$|S_k(P, \tilde{\Pi})| \gtrsim |V_k|^{k+1} \frac{|\tilde{\Pi}|^{k^2-2k-2}}{|P|^k |\tilde{I}|^{k^2-k-2}}. \quad (38)$$

where the set V_k is as defined in equation (34).

Remark 4.6. Heuristically, one can easily compute this lower bound on $|S_k(P, \Pi)|$. Since a point from P and a k -plane from Π are incident with probability $\frac{|I|}{|P||\Pi|}$, and a simplex consists of $(k+1)(k+2)$ incidences amongst $(k+2)$ points and $(k+2)$ k -planes, it follows that

$$|S_k(P, \Pi)| \gtrsim |P|^{k+2} |\Pi|^{k+2} \left(\frac{|I|}{|P||\Pi|} \right)^{(k+1)(k+2)}$$

under the unrealistic assumption that all events of point-plane incidence are independent. To validate this heuristic rigorously, we shall make several refinements in order to ensure a certain amount of geometric regularity regarding the

distribution of points within each k -plane, hence creating a logarithmic loss in the end result. If a new procedure for demonstrating the heuristically obvious lower bound can be established, then the incidence bound might be slightly improved (i.e. we can replace the symbol \gtrsim with \lesssim).

The proof of this lemma will be a somewhat complicated induction on k . For the sake of clarity, it will be useful to briefly outline the proof before proceeding.

Definition 4.7. Given a collection of points P and k -planes $\tilde{\Pi}$, let $C1(k, P, \tilde{\Pi})$ denote the statement that the conclusion of lemma 4.5 holds.

Definition 4.8. Given a collection of points P and k -planes $\tilde{\Pi}$, let $C2(k, P, \tilde{\Pi})$ denote the statement that

$$|S_k(P, \tilde{\Pi})| \gtrsim \frac{|I|^{(k+1)(k+2)}}{|P|^{k(k+2)} |\tilde{\Pi}|^{k(k+2)}}. \quad (39)$$

With this terminology established, we may describe the inductive procedure. First, we will prove that $C2(0, R, \Sigma)$ is true for any collection of points R and “affine 0-planes” (i.e. points) in an ambient space of any dimension. Though the case $k = 0$ is a bit absurd, it will work (formally) as the base of the induction. The reader who is unsatisfied with this may choose to refer to the arguments found in [17], and begin the induction at $k = 1$. Next, we make the following essentially trivial observation:

Lemma 4.9. *Let k be an integer, $k \geq 1$. Given a collection of points R , and a collection of k -planes Σ (living in an ambient space of any dimension strictly larger than k), we have the following implication:*

$$\{H1(k, R, \Sigma) \ \& \ H2(k, R, \Sigma)\} \Rightarrow \{C1(k, R, \Sigma) \Rightarrow C2(k, R, \Sigma)\}. \quad (40)$$

Proof. The proof of this sub-lemma is simply a computation making use of bounds we have already demonstrated. Use hypotheses $H1(k, R, \Sigma)$ and $H2(k, R, \Sigma)$ to assume all of the regularity conditions needed for the estimates

$$|V_k(R, \Sigma)| \gtrsim \frac{|\tilde{I}_k|^2}{R^k} \quad (41)$$

$$|I_k(R, \Sigma)| \gtrsim \frac{|I(R, \Sigma)|^k}{|\Sigma|^{k-1}}. \quad (42)$$

where the sets $V_k(R, \Sigma)$ and $I_k(R, \Sigma)$ are defined as in equations (34) and (31) (respectively).

Then, simply insert these bounds into equation (38). □

The final ingredient needed to prove lemma 4.5 is a statement the $C2(k-1, P_{\pi_0, x}, \Sigma_{\pi_0, x}) \Rightarrow C1(k, P, \tilde{\Pi})$ (under suitable hypotheses), where the sets $P_{\pi_0, x}$ and $\Sigma_{\pi_0, x}$ are collections of points and $(k-1)$ -planes (respectively) to be defined later. With this argument in place, the proof will then, schematically, look like the following:

$$C2(0) \Rightarrow C1(1) \Rightarrow C2(1) \Rightarrow C1(2) \Rightarrow C2(2) \cdots$$

Unfortunately, there are two technicalities we must dispense with before beginning with the above program. The method described will not work unless we can ensure that simplices formed at every level of the induction are non-degenerate. In order to avoid potential non-degeneracies, we must show that the hypotheses $H1(k, P, \tilde{\Pi})$ and $H2(k, P, \tilde{\Pi})$ imply a new pair of hypotheses $H1(k-1, Q, \Sigma)$ and $H2(k-1, Q, \Sigma)$ for suitably defined sets of points Q , and $(k-1)$ -planes Σ (and so on for $(k-2)$ -planes, $(k-3)$ -planes, etc...). In order to simplify the exposition, we will assume, for the moment, that the regularity hypotheses hold at every level of the induction. Then, in Section 5, we will verify that this is indeed the case.

Proof of Lemma 4.5. To begin the proof, we let R denote a collection of points, and Σ a collection of affine 0-planes in some ambient space (of any finite dimension ≥ 1). We wish to show that the conclusion $C2(0, R, \Sigma)$ holds; this essentially means that we must show

$$|S_0(R, \Sigma)| \gtrsim |I(R, \Sigma)|^2. \quad (43)$$

This estimate, however, is trivial after observing that $I(R, \Sigma) = R \cap \Sigma$, and $S_0(R, \Sigma)$ denotes the set of all line segments with distinct endpoints in the set $I(R, \Sigma)$.

In order to begin the inductive part of the proof, we will need to introduce a family of $(k-1)$ -planes. Given a set of points P , and k -planes $\tilde{\Pi}$ satisfying $H1(k, P, \tilde{\Pi})$ and $H2(k, P, \tilde{\Pi})$, fix an element $(\pi_0, x) \in D$ (as defined in equation (37)). For this particular pair, we define a family of $(k-1)$ -planes $\Sigma_{\pi_0, x}$ defined as:

$$\Sigma_{\pi_0, x} := \left\{ \sigma \subset \pi_0 : \begin{array}{l} \exists (\pi, p_1, \dots, p_k) \in \tilde{I}_k \text{ for which} \\ \sigma = [p_1, \dots, p_k], ((\pi_0, \pi, p_1, \dots, p_k), x) \in V_{k, p} \end{array} \right\} \quad (44)$$

Also, we need to identify a class of points

$$P_{\pi_0, x} := P \cap \pi_0 \quad (45)$$

which will interact with the set $\Sigma_{\pi_0, x}$. In Section 5 we will verify that there exists a set $\tilde{D}(P, \tilde{\Pi}) \subset D(P, \tilde{\Pi})$ such that $|\tilde{D}(P, \tilde{\Pi})| \geq \frac{1}{2}|D(P, \tilde{\Pi})|$ and the hypotheses $H1(k-1, P_{\pi_0, x}, \Sigma_{\pi_0, x})$ and $H2(k-1, P_{\pi_0, x}, \Sigma_{\pi_0, x})$ are true for each pair

$(\pi_0, x) \in \tilde{D}(P, \tilde{\Pi})$. Therefore, we may assume that the configurations of points and planes $(P_{\pi_0, x}, \Sigma_{\pi_0, x})$ induced from this large subclass enjoy the same regularity properties as $(P, \tilde{\Pi})$.

Next, we observe that the set of $(k+1)$ -simplices in S_k which contain the k -plane π_0 as a face and the point x as a vertex are in one to one correspondence with the set of k -simplices contained in π_0 whose faces belong to $\Sigma_{\pi_0, x}$ and vertices belong to $P_{\pi_0, x}$. In other words, each such $(k+1)$ -simplex is seen as the cone of some k -simplex contained in π_0 . So, once we have a lower bound on the number of k -simplices contained in π_0 , we will obtain a lower bound on the number of $(k+1)$ -simplices containing π_0 and x .

We formalize this as follows: Let $J = I(P_{\pi_0, x}, \Sigma_{\pi_0, x})$ denote the incidence set arising from the collection of points P_{π_0} and the collection of $(k-1)$ -planes $\Sigma_{\pi_0, x}$. Also, define the set

$$J_{k, \pi_0, x} := \left\{ (\sigma, p_1, \dots, p_k) \in \Sigma_{\pi_0, x} \times P_{\pi_0, x}^k : \begin{array}{l} p_i \in \sigma \text{ for all } i, \\ \dim[p_1, \dots, p_k] = k-1 \end{array} \right\}$$

Next, we observe that (by the construction of the sets $\Sigma_{\pi_0, x}$ and $P_{\pi_0, x}$) there exists a one-to-one correspondence between the set $J_{k, \pi_0, x}$ and the set $\{(\pi, p_1, \dots, p_k) \in \tilde{I}_k : ((\pi_0, \pi, p_1, \dots, p_k), x) \in V_{k, p}\}$. So, we have the following:

$$\begin{aligned} \frac{|J|^k}{|\Sigma_{\pi_0, x}|^{k-1}} &\sim |J_{k, \pi_0, x}| \\ &= |\{(\pi, p_1, \dots, p_k) \in \tilde{I}_k : ((\pi_0, \pi, p_1, \dots, p_k), x) \in V_{k, p}\}| \\ &\gtrsim |V_k| \frac{|\tilde{I}|}{|\tilde{\Pi}| |D(P, \tilde{\Pi})|}. \end{aligned}$$

Now we invoke the inductive hypothesis: letting $S_{k-1, \pi_0, x}$ denote the set of k -simplices contained in π_0 with faces from $\Sigma_{\pi_0, x}$ and vertices from $P_{\pi_0, x}$, we assume the statement C2($k-1, P_{\pi_0, x}, \Sigma_{\pi_0, x}$) is true. Hence,

$$|S_{k-1, \pi_0, x}| \gtrsim \frac{|J|^{k(k+1)}}{|\Sigma_{\pi_0, x}|^{(k-1)(k+1)} |P_{\pi_0, x}|^{(k-1)(k+1)}}. \quad (46)$$

Next, we use the previous computation to reinterpret this lower bound as:

$$|S_{k-1, \pi_0, x}| \gtrsim \left(|V_k| \frac{|\tilde{I}|}{|\tilde{\Pi}| |D(P, \tilde{\Pi})|} \right)^{k+1} \frac{1}{|P_{\pi_0, x}|^{(k-1)(k+1)}} \quad (47)$$

Now let $S_{k, \pi_0, x}$ denote the set of $(k+1)$ -simplices which contain the k -plane π_0 as a face and the point x as a vertex. Recall that every simplex in $S_{k, \pi_0, x}$ is the

cone of a simplex from $S_{k-1, \pi_0, x}$. So,

$$|S_{k, \pi_0, x}| \gtrsim \left(|V_k| \frac{|\tilde{I}|}{|\tilde{\Pi}| |D(P, \tilde{\Pi})|} \right)^{k+1} \frac{1}{|P_{\pi_0, x}|^{(k-1)(k+1)}}. \quad (48)$$

Next observe that for any π_0 , $|P_{\pi_0, x}| \approx \frac{|\tilde{I}|}{|\tilde{\Pi}|}$. So, we have

$$|S_{k, \pi_0, x}| \gtrsim \left(|V_k| \frac{|\tilde{I}|}{|\tilde{\Pi}| |D(P, \tilde{\Pi})|} \right)^{k+1} \left(\frac{|\tilde{\Pi}|}{|\tilde{I}|} \right)^{(k-1)(k+1)}. \quad (49)$$

To finish the computation, simply observe that this bound holds for any pair $(\pi_0, x) \in D(P, \tilde{\Pi})$. So, simply multiply the previous bound by $|D(P, \tilde{\Pi})|$, and then insert the trivial bound $|D(P, \tilde{\Pi})| \leq |P| |\tilde{\Pi}|$. This finishes the proof of the lemma. \square

5 Simplex Construction Part Two: Regularity Hypotheses

The material discussed in this section should essentially be viewed as an appendix to the previous section. In particular, we wish to verify that the simplices constructed in the previous section are not degenerate in the sense that their faces (edges, vertices, hyper-edges, etc.) do not collapse onto each other. This nondegeneracy can be demonstrated by showing that the hypotheses $H1(r, Q, \Sigma)$ and $H2(r, Q, \Sigma)$ hold for most of the arrangements of points and r -planes (Q, Σ) appearing throughout the induction.

Before beginning the process of verifying these hypotheses, it will be useful to formally describe the construction of the collections of points and r -planes we shall be working with. The procedure is virtually identical to the one seen in the previous section for constructing the collections $P_{\pi_0, x}$ and $\Sigma_{\pi_0, x}$. Let Q be a collection of points and Σ a collection of r -planes embedded in some $(r+1)$ -dimensional space, and assume that the hypotheses $H1(r, Q, \Sigma)$ and $H2(r, Q, \Sigma)$ have been verified. Working with these sets, we may (as we have already done with k -planes) construct the set $V_{r, p}(Q, \Sigma)$:

$$V_{r, p}(Q, \Sigma) := \left\{ \begin{array}{l} (\sigma_0, \sigma, p_1, \dots, p_r, x) \\ \in \Sigma^2 \times Q^{r+1} \end{array} : \begin{array}{l} [p_1, \dots, p_r] \subset \sigma_0 \cap \sigma \\ \dim[p_1, \dots, p_r] = r-1 \\ x \in \sigma \setminus \sigma_0, \sigma \neq \sigma_0 \\ |[p_1, \dots, p_r] \cap Q| \geq \frac{|I(Q, \Sigma)|}{10|\Sigma||F|} \end{array} \right\}. \quad (50)$$

Since we're assuming the hypotheses $H1(r, Q, \Sigma)$ and $H2(r, Q, \Sigma)$ hold, we may compute a lower bound for the size of this set as

$$|V_{r,p}| \gtrsim \frac{|I(Q, \Sigma)|^{2r+1}}{|Q|^r |\Sigma|^{2r-1}}. \quad (51)$$

Next, we define a function f_r on the collection of all pairs $(\sigma_0, x) \in \Sigma \times Q$ such that $x \notin \sigma_0$ as

$$f_r(\sigma_0, x) := |\{(\sigma, p_1, \dots, p_r) \in \Sigma \times Q^r : (\sigma_0, \sigma, p_1, \dots, p_r, x) \in V_{r,p}(Q, \Sigma)\}|. \quad (52)$$

Clearly, we have

$$\sum_{(\sigma_0, x)} f_r(\sigma_0, x) \sim |V_{r,p}(Q, \Sigma)|.$$

So, we may once again use a dyadic pigeonholing argument to find a nice collection $D(Q, \Sigma) \subset Q \times \Sigma$ such that

$$f_r(\sigma_0, x) \approx \frac{|V_{r,p}(Q, \Sigma)|}{|D(Q, \Sigma)|} \text{ for all } (\sigma_0, x) \in D(Q, \Sigma). \quad (53)$$

Now, we are in a position to define the relevant collections of $(r-1)$ -planes necessary for the construction. Given a pair $(\sigma_0, x) \in D(Q, \Sigma)$, We define a collection of $(r-1)$ -planes embedded in σ_0 as

$$\Gamma_{\sigma_0, x} := \left\{ \gamma \subset \sigma_0 : \begin{array}{l} \exists (\sigma, p_1, \dots, p_r) \in \sigma \text{ for which } \gamma = [p_1, \dots, p_r], \\ ((\sigma_0, \sigma, p_1, \dots, p_r), x) \in V_{r,p}(Q, \Sigma) \end{array} \right\}. \quad (54)$$

Also, we may simply define the associated collection of points as

$$Q_{\sigma_0, x} = Q \cap \sigma_0. \quad (55)$$

As it turns out, the verification of $H2$ at each level of the induction is far more elementary than the verification of $H1$; in fact, $H2$ essentially holds by definition. We state this formally as follows:

Lemma 5.1. *Let $(P, \tilde{\Pi})$ be an arrangement of points and k -planes embedded in some ambient space as described in equations (23) and (22). Given a pair $(\pi_0, x) \in D(P, \tilde{\Pi})$, define collections of points $P_{\pi_0, x}$ and $(k-1)$ -planes $\Sigma_{\pi_0, x}$ embedded in π_0 as defined in equations (45) and (44) respectively. Then, we have the following:*

$$H2(k, P, \tilde{\Pi}) \Rightarrow H2(k-1, P_{\pi_0, x}, \Sigma_{\pi_0, x}) \quad (56)$$

for each of the pairs (x, π_0) . Furthermore, this regularity holds at all lower levels of the induction in the sense that (using the notation already established in this section)

$$H2(r, Q, \Sigma) \Rightarrow H2(r-1, Q_{\pi_0, x}, \Gamma_{\sigma_0, x}) \quad (57)$$

for each of the pairs (σ_0, x) .

Proof. First, let us recall the definitions of the sets $P_{\pi_0, x}$ and $\Sigma_{\pi_0, x}$:

$$P_{\pi_0, x} = P \cap \pi_0, \\ \Sigma_{\pi_0, x} = \left\{ \sigma \subset \pi_0 : \begin{array}{l} \exists (\pi, p_1, \dots, p_k) \in \tilde{I}_k \text{ for which} \\ \sigma = [p_1, \dots, p_k], ((\pi_0, \pi, p_1, \dots, p_k), x) \in V_{k, p} \end{array} \right\}.$$

Next, let us recall the definition of \tilde{I}_k :

$$\tilde{I}_k = \left\{ (\pi, p_1, \dots, p_k) \in \tilde{\Pi} \times P^k : \begin{array}{l} \dim[p_1, \dots, p_k] = k - 1, \\ |[p_1, \dots, p_k] \cap P| \geq \frac{|\tilde{I}|}{10|\tilde{\Pi}||F|} \end{array} \right\}.$$

Suppose that $H2(k-1, P_{\pi_0, x}, \Sigma_{\pi_0, x})$ failed for some pair $(\pi_0, x) \in D$; this would imply

$$|I(P_{\pi_0, x}, \Sigma_{\pi_0, x})| \lesssim |\Sigma_{\pi_0, x}| |F|^{k-2}. \quad (58)$$

This, however, is a contradiction to the assumption that $H2(k, P, \Pi)$ holds, as the above definitions imply the following string of inequalities:

$$\frac{|\tilde{I}|}{10|\tilde{\Pi}||F|} \lesssim \frac{|I(P_{\pi_0, x}, \Sigma_{\pi_0, x})|}{|\Sigma_{\pi_0, x}|} \lesssim |F|^{k-2}$$

or,

$$|\tilde{I}| \lesssim |\tilde{\Pi}| |F|^{k-1}. \quad (59)$$

We omit the proof for the remaining levels of the induction, as it is virtually identical to what we have already shown. \square

In order to show that property H1 is inherited at all stages of the induction, one needs to do a bit more work. The reason for this is that property H1 does not simply pass freely from level k to level $k-1$ by definition (as was essentially the case for property H2). Loosely speaking, the statement $H1(k, P, \tilde{\Pi})$ states that most maximal-rank k -tuples of points in the set P are incident to many k -planes from the set $\tilde{\Pi}$ (in this case $k = 1$, this general property is usually referred to as “bilinearity”). To illustrate the difficulty, consider the case $k = 2$, $n \geq 4$. Choose a pair $(\pi_0, x) \in D$, and construct the sets $P_{\pi_0, x}$ and $\Sigma_{\pi_0, x}$ in the usual fashion. We would like to say that the hypothesis $H1(1, P_{\pi_0, x}, \Sigma_{\pi_0, x})$ is satisfied, or (more informally) that most of the points in $P_{\pi_0, x}$ are incident to many lines from $\Sigma_{\pi_0, x}$. A natural attempt to prove such a statement is to observe that, by property $H1(2, P, \tilde{\Pi})$, most pairs of points in P are incident to many planes from $\tilde{\Pi}$. In particular, given a point $p \in P_{\pi_0, x} \subset P$, there are many planes incident to the pair $(p, x) \in P^2$. Unfortunately, since $n \geq 4$, these 2-planes are not obligated to intersect π_0 in a line, hence not contributing to the set $\Sigma_{\pi_0, x}$. Therefore, one must come up with a different means by which to verify $H1(1, P_{\pi_0, x}, \Sigma_{\pi_0, x})$.

Lemma 5.2. *Let $(P, \tilde{\Pi})$ be an arrangement of points and k -planes embedded in some ambient space as described in equations (23) and (22). Given a pair $(\pi_0, x) \in D(P, \tilde{\Pi})$, define collections of points $P_{\pi_0, x}$ and $(k-1)$ -planes $\Sigma_{\pi_0, x}$ embedded in π_0 as defined in equations (45) and (44) respectively. Then, we have the following:*

$$H1(k, P, \tilde{\Pi}) \Rightarrow H1(k-1, P_{\pi_0, x}, \Sigma_{\pi_0, x}) \quad (60)$$

for at least half of the pairs (π_0, x) . Furthermore, this regularity holds at all lower levels of the induction in the sense that (using the notation already established in this section)

$$H1(r, Q, \Sigma) \Rightarrow H1(r-1, Q_{\sigma_0, x}, \Gamma_{\sigma_0, x}) \quad (61)$$

for at least half of the pairs (σ_0, x) .

Proof. Given $(\pi_0, x) \in D(P, \tilde{\Pi})$, define sets $P_{\pi_0, x}$ and $\Sigma_{\pi_0, x}$ as above. Assuming that $H1(k-1, P_{\pi_0, x}, \Sigma_{\pi_0, x})$ fails when $(\pi_0, x) \in D'(P, \tilde{\Pi})$, where $D'(P, \tilde{\Pi}) \subset D(P, \tilde{\Pi})$ and $|D'(P, \tilde{\Pi})| \geq \frac{1}{2}|D(P, \tilde{\Pi})|$, we will obtain upper and lower bounds for the size of the set $V_{k,p}(P, \tilde{\Pi})$ as defined in equation (35). For convenience, let us recall this definition:

$$V_{k,p} = \{((\pi_0, \pi, p_1, \dots, p_k), x) \in V_k \times P : x \in P \cap (\pi \setminus [p_1, \dots, p_k])\}$$

Assuming $H1(k, P, \tilde{\Pi})$ and $H2(k, P, \tilde{\Pi})$ hold, we may use the computations from the previous section, and obtain a lower bound of

$$|V_{k,p}| \gtrsim \frac{|\tilde{I}|^{2k+1}}{|P|^k |\tilde{\Pi}|^{2k-1}}. \quad (62)$$

Before proceeding, we will refine the set $V_{k,p}$ to the following slightly smaller subset

$$V'_{k,p} = \{((\pi_0, \pi, p_1, \dots, p_k), x) \in V_{k,p} : (x, \pi_0) \in D'(P, \tilde{\Pi})\}. \quad (63)$$

Since there is a uniform (throughout the set $D(P, \tilde{\Pi})$) lower bound on the number of objects of the form $(\pi, p_1, \dots, p_k) \in I_k(P, \tilde{\Pi})$ such that $((\pi_0, \pi, p_1, \dots, p_k), x) \in V_{k,p}$, and $|D'(P, \tilde{\Pi})| \geq \frac{1}{2}|D(P, \tilde{\Pi})|$, we have $|V'_{k,p}(P, \tilde{\Pi})| \sim |V_{k,p}(P, \tilde{\Pi})|$.

Next, we compute an upper bound for the size of $V'_{k,p}(P, \tilde{\Pi})$ in two slightly different ways. First of all, there are $\lesssim |P||\tilde{\Pi}|$ choices for the pair (π_0, x) . Once this pair has been fixed, we make use of the estimate $|P \cap \pi_0| \approx \frac{|\tilde{I}|}{|\tilde{\Pi}|}$, and observe that there are $\approx \left(\frac{|\tilde{I}|}{|\tilde{\Pi}|}\right)^{k-1}$ choices for the $(k-1)$ -tuple (p_1, \dots, p_{k-1}) . Next, assuming that $H1(k-1, P_{\pi_0, x}, \Sigma_{\pi_0, x})$ fails, it follows that there are at most $O(1)$

k -planes from $\tilde{\Pi}$ which contain the $(k-1)$ -tuple (p_1, \dots, p_{k-1}) and contribute a $(k-1)$ -plane to the set $\Sigma_{\pi_0, x}$. This means that there are at most $\sim |F|^{k-1}$ choices for the remaining point p_k . Putting this all together, we have an estimate of

$$|V'_{k,p}| \lesssim |P| |\tilde{\Pi}| \left(\frac{|\tilde{I}|}{|\tilde{\Pi}|} \right)^{k-1} |F|^{k-1}. \quad (64)$$

Also, we will need to make another, seemingly cruder estimate of the same set. This estimate is obtained in the same manner as equation (64), except we use the trivial estimate $|P \cap \pi_0| \leq |F|^k$ in place of the estimate $|P \cap \pi_0| \approx \frac{|\tilde{I}|}{|\tilde{\Pi}|}$. This yields an upper bound of

$$|V'_{k,p}| \lesssim |P| |\tilde{\Pi}| |F|^{k^2-1}. \quad (65)$$

Combining the upper bounds (64) and (65) with the lower bound (62), we obtain the incidence bounds

$$|I| \lesssim |P|^{\frac{k+1}{k+2}} |\tilde{\Pi}|^{\frac{k+1}{k+2}} |F|^{\frac{k-1}{k+2}} \quad (66)$$

and

$$|I| \lesssim |P|^{\frac{k+1}{2k+1}} |\tilde{\Pi}|^{\frac{2k}{2k+1}} |F|^{\frac{k^2-1}{2k+1}} \quad (67)$$

respectively.

Next, for each integer k , we define a quantity $\alpha(k)$ as

$$\alpha(k) = \frac{k^3 + k^2 - 4k - 4}{k^3 + k^2 - 2}. \quad (68)$$

While this quantity may seem a bit odd, the reader will notice that it satisfies the inequalities $0 \leq \alpha(k) \leq 1$ for every k , and (after a great deal of tedious algebra), the following miracle occurs:

$$\begin{aligned} |I| &\lesssim \left(|P|^{\frac{k+1}{k+2}} |\tilde{\Pi}|^{\frac{k+1}{k+2}} |F|^{\frac{k-1}{k+2}} \right)^{\alpha(k)} \left(|P|^{\frac{k+1}{2k+1}} |\tilde{\Pi}|^{\frac{2k}{2k+1}} |F|^{\frac{k^2-1}{2k+1}} \right)^{1-\alpha(k)} \\ &= |P|^{\frac{k(k+1)}{k^2+2k+2}} |\tilde{\Pi}|^{\frac{k^2+k+2}{k^2+2k+2}} |F|^{\frac{k(k+1)}{k^2+2k+2}}. \end{aligned}$$

As this is precisely the incidence bound that we are trying to demonstrate, it follows that we may assume $\text{H1}(k-1, P_{\pi_0, x}, \Sigma_{\pi_0, x})$ holds for most pairs $(\pi_0, x) \in D$.

Next, given an integer $1 \leq r \leq k-2$, we assume that collections of points Q and $(k-r+1)$ -planes Σ (as constructed at the start of this section) satisfy hypothesis $\text{H1}(k-r+1, Q, \Sigma)$:

$$|I(Q, \Sigma)| \gg |Q| |\Sigma|^{\frac{k-r}{k-r+1}}. \quad (69)$$

Suppose that $H1(k-r, Q_{\sigma_0, x}, \Gamma_{\sigma_0, x})$ fails when $(\sigma_0, x) \in D'(Q, \Sigma)$ for some set $D'(Q, \Sigma) \subset D(Q, \Sigma)$ where $|D'(Q, \Sigma)| \geq \frac{1}{2}|D(Q, \Sigma)|$. then, we have the following:

$$|I(Q_{\sigma_0, x}, \Gamma_{\sigma_0, x})| \lesssim |Q_{\sigma_0, x}| |\Gamma_{\sigma_0, x}|^{\frac{k-r-1}{k-r}} \text{ for } (\sigma_0, x) \in D'(Q, \Sigma). \quad (70)$$

In order to show that one may assume that this situation doesn't occur, we will proceed in a manner similar to the way we addressed the $r = 1$ case, however we will be counting objects which are a bit more complicated than elements of the set $V_{k, p}$.

Definition 5.3. Given integers k and l , we define a (k, l) -chain to be a $(k+2)$ -tuple of points $(p_0, \dots, p_{k+1}) \in P^{k+2}$, and an l tuple of k -planes $(\pi_0, \dots, \pi_{l-1}) \in \tilde{\Pi}^l$ such that (p_0, \dots, p_{k+1}) spans a $(k+1)$ -dimensional space, and for any positive integer $m \leq l$, and any choice of m k -planes from the l -tuple $(\pi_0, \dots, \pi_{l-1})$, the space $\pi_{i_1} \cap \dots \cap \pi_{i_m}$ is $(k-m+1)$ -dimensional, and spanned by some $(k-m+2)$ -tuple of points from $(k+2)$ -tuple (p_0, \dots, p_{k+1}) . The set of all (k, l) -chains constructed from the sets P and $\tilde{\Pi}$ will be denoted $C_{k, l}(P, \tilde{\Pi})$.

Remark 5.4. As the definition of a (k, l) -chain is somewhat complicated, one might wish to regard them as $(k+1)$ -simplices (with faces from $\tilde{\Pi}$ and vertices from P) which are missing $k-l+2$ of their faces, but none of their vertices.

This construction is motivated by the fact that when $l = r+1$, the intersection of the $r+1$ k -planes in any $(k, r+1)$ chain in the set $C_{k, r+1}(P, \tilde{\Pi})$ will be a $(k-r)$ -plane from the set $\Gamma_{\sigma_0, x}$. Therefore, if we assume that equation (70) holds, we will be able to demonstrate an upper bound for $|C_{k, r+1}|$. Then, after computing a corresponding lower bound (mostly by means of Cauchy-Schwarz and pigeonholing techniques), we will be able to show that the incidence bound (25) holds.

Our first task will be to compute a lower bound for a certain large subclass of $C_{k, r+1}(P, \tilde{\Pi})$. Unfortunately, this computation is much more complicated than it was in the case $r = 1$; the reason for this is that Cauchy-Schwarz alone does not seem to be powerful enough to count (k, l) -chains when $l \geq 3$. Therefore, we shall use a “coning procedure”, much like was seen in the previous section for counting simplices, in order to estimate $|C_{k, r+1}(P, \tilde{\Pi})|$ from below. The basic scheme for this construction will be to realize any $(k, r+1)$ -chain as the cone of a $(k-1, r)$ -chain, which in turn is the cone of a $(k-2, r-1)$ -chain, and so on (see figure 5). Eventually, elements of the set $C_{k, r+1}(P, \tilde{\Pi})$ will be described as hypercones of $(k-r+1, 2)$ -chains. Since Cauchy-Schwarz is useful for estimating chains of length 2, we will arrive at the desired lower bound for $|C_{k, r+1}|$, by estimating the size of a certain class of $(k-r+1, 2)$ -chains, and then making a few observations.

This computation will require a bit of extra notation. As was shown in the

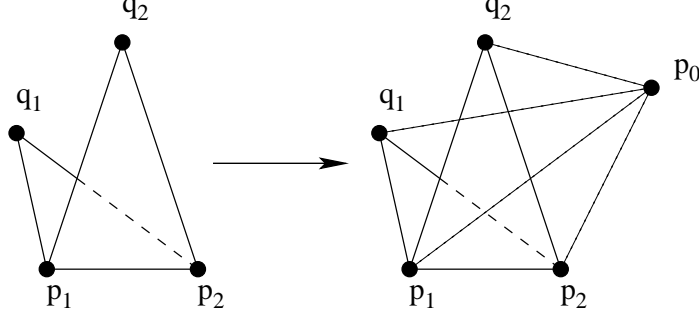


Figure 2: This figure illustrates the realization of a $(3,3)$ -chain as the cone of a $(2,2)$ -chain. The three 3-planes in the $(3,3)$ -chain on the right are $\pi_0 = [p_1, p_2, q_1, q_2]$, $\pi_1 = [p_1, p_2, p_0, q_1]$ and $\pi_2 = [p_1, p_2, p_0, q_2]$.

previous section, it is possible to find a collection of disjoint pairs of points and k -planes $D(P, \tilde{\Pi}) \subset P \times \tilde{\Pi}$ such that for each $(p_0, \pi_0) \in D(P, \tilde{\Pi})$ one has

$$f(p_0, \pi_0) \gtrsim \frac{|I|^{2k+1}}{|P|^k |\tilde{\Pi}|^{2k-1} |D(P, \tilde{\Pi})|}, \quad (71)$$

where the function f is defined (for pairs $(p_0, \pi_0) \in P \times \tilde{\Pi}$) as

$$f(p_0, \pi_0) = |\{(\pi, p_1, \dots, p_k) \in \tilde{I}_k : ((\pi_0, \pi, p_1, \dots, p_k), p_0) \in V_{k,p}(P, \tilde{\Pi})\}|. \quad (72)$$

So, for each $(p_0, \pi_0) \in D(P, \tilde{\Pi})$, one can define families $Q_{p_0, \pi_0}^{(k-1)}$ and $\Sigma_{p_0, \pi_0}^{(k-1)}$ of points and $(k-1)$ -planes contained in π_0 . By definition, these collections satisfy the following estimates:

$$|Q_{p_0, \pi_0}^{(k-1)}| \approx \frac{|\tilde{I}|}{|\tilde{\Pi}|} \quad (73)$$

$$|I_k(Q_{p_0, \pi_0}^{(k-1)}, \Sigma_{p_0, \pi_0}^{(k-1)})| \gtrsim \frac{|\tilde{I}|^{2k+1}}{|P|^k |\tilde{\Pi}|^{2k-1} |D(P, \tilde{\Pi})|}. \quad (74)$$

Next, we observe that any element of the set $C_{k,r+1}(P, \tilde{\Pi})$ which contains the point p_0 as a vertex and the k -plane π_0 as a face must be the cone of a unique element of the set $C_{k-1,r}(Q_{p_0, \pi_0}^{(k-1)}, \Sigma_{p_0, \pi_0}^{(k-1)})$. This observation allows us to conclude that

$$|C_{k,r+1}(P, \tilde{\Pi})| \gtrsim \sum_{(p_0, \pi_0) \in D(P, \tilde{\Pi})} |C_{k-1,r}(Q_{p_0, \pi_0}^{(k-1)}, \Sigma_{p_0, \pi_0}^{(k-1)})|.$$

So, by a simple application of the pigeonhole principle, we may choose a single value $(p_0, \pi_0) \in D(P, \tilde{\Pi})$ such that

$$|C_{k,r+1}(P, \tilde{\Pi})| \gtrsim |C_{k-1,r}(Q_{p_0, \pi_0}^{(k-1)}, \Sigma_{p_0, \pi_0}^{(k-1)})| |D(P, \tilde{\Pi})|. \quad (75)$$

Therefore, we have reduced our problem to that of estimating the size of the sets $C_{k-1,r}(Q_{p_0,\pi_0}^{(k-1)}, \Sigma_{p_0,\pi_0}^{(k-1)})$. At this point, the procedure repeats itself, this time with the collections $Q_{p_0,\pi_0}^{(k-1)}$ and $\Sigma_{p_0,\pi_0}^{(k-1)}$ of points and $(k-1)$ -planes.

Remark 5.5. Since these estimates hold uniformly for a specifically chosen (via the pigeonhole principle) pair $(p_0, \pi_0) \in D(P, \tilde{\Pi})$, we will no longer use these parameters as subscripts when describing induced sets of points or $(k-1)$ -planes. For example, the set $\Sigma^{(k-1)}$ will be used to denote an induced family of $(k-1)$ -planes relative to some generic $(p_0, \pi_0) \in D(P, \tilde{\Pi})$. This convention is purely in the interest of simplifying notation, and will be used for analogous constructions to appear later in the proof. Furthermore, the sets $\Gamma_{\sigma_0,x}$ referred to in equation (70) will henceforth be denoted $\Sigma^{(k-r)}$.

In general, once the sets $Q^{(k-l)}$ and $\Sigma^{(k-l)}$ have been constructed (for some integer $1 \leq l \leq r-2$), one can choose a collection $D(Q^{(k-l)}, \Sigma^{(k-l)}) \subset Q^{(k-l)} \times \Sigma^{(k-l)}$ of disjoint pairs of points and $(k-l)$ -planes, and use this collection to define (as was described at the beginning of this section) to create families $Q^{(k-l-1)}$ and $\Sigma^{(k-l-1)}$ of points and $(k-l-1)$ -planes which satisfy the estimates

$$|Q^{(k-l-1)}| \approx \frac{|I(Q^{(k-l)}, \Sigma^{(k-l)})|}{|\Sigma^{(k-l)}|} \quad (76)$$

$$|I_{k-l}(Q^{(k-l-1)}, \Sigma^{(k-l-1)})| \gtrsim \frac{|I(Q^{(k-l)}, \Sigma^{(k-l)})|^{2(k-l)+1}}{|Q^{(k-l)}|^{k-l} |\Sigma^{(k-l)}|^{2(k-l)-1} |D(Q^{(k-l)}, \Sigma^{(k-l)})|}. \quad (77)$$

Furthermore, by identifying elements of the sets $C_{k-l,r+1-l}(Q^{(k-l)}, \Sigma^{(k-l)})$ and $C_{k-l-1,r-l}(Q^{(k-l-1)}, \Sigma^{(k-l-1)})$, we have the estimate

$$|C_{k-l,r+1-l}(Q^{(k-l)}, \Sigma^{(k-l)})| \gtrsim |C_{k-l-1,r-l}(Q^{(k-l-1)}, \Sigma^{(k-l-1)})| |D(Q^{(k-l)}, \Sigma^{(k-l)})|. \quad (78)$$

Since these observations hold for any integer $1 \leq l \leq r-2$, we arrive at the following conclusion:

$$|C_{k,r+1}(P, \tilde{\Pi})| \gtrsim |C_{k-r+1,2}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})| |D(P, \tilde{\Pi})| \prod_{j=k-r+2}^{k-1} |D(Q^{(j)}, \Sigma^{(j)})|. \quad (79)$$

At this point, since we have reduced to problem to estimating the size of a family of $(k-r+1, 2)$ -chains, we may use Cauchy-Schwarz. The construction of elements of the set $C_{k+r-1,2}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})$ is virtually identical to the construction of the sets $V_{2,p}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})$, except that we add an extra point to each of the $(k-r+1)$ -planes, rather than just one of them. Therefore,

Cauchy-Schwarz yields an estimate of

$$\begin{aligned}
|C_{k+r-1,2}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})| &\gtrsim \frac{|I_{k-r+1}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|^2}{|Q^{(k-r+1)}|^{k-r+1}} \\
&\frac{|I(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|^2}{|\Sigma^{(k-r+1)}|^2} \\
&\gtrsim \frac{|I(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|^{2(k-r+2)}}{|Q^{(k-r+1)}|^{k-r+1} |\Sigma^{(k-r+1)}|^{2(k-r+1)}}. \quad (80)
\end{aligned}$$

Furthermore, since we are assuming (by induction) that the hypothesis H1($k-r+1, Q^{(k-r+1)}, \Sigma^{(k-r+1)}$) holds, it follows that these $(k-r+1, 2)$ -chains are non-degenerate (i.e. the two $(k-r+1)$ -planes contained in each are distinct). Explicitly, the degenerate $(k-r+1)$ -chains which we would like to discard are simply elements of the set $I_{k-r+1}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})$ equipped with an additional pair of points, so it has size $\approx |I_{k-r+1}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})| \frac{|I_{k-r+1}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|^2}{|\Sigma^{(k-r+1)}|^2}$. Applying the hypothesis H1($k-r+1, Q^{(k-r+1)}, \Sigma^{(k-r+1)}$) yields

$$\begin{aligned}
|C_{k+r-1,2}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})| &\gtrsim \frac{|I_{k-r+1}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|^2}{|Q^{(k-r+1)}|^{k-r+1}} \\
&\frac{|I(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|^2}{|\Sigma^{(k-r+1)}|^2} \\
&\gtrsim \frac{|I_{k-r+1}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|^{k-r+1}}{|Q^{(k-r+1)}|^{k-r+1} |\Sigma^{(k-r+1)}|^{k-r}} \\
&\frac{|I_{k-r+1}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|}{|I(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|^2} \\
&\frac{|I(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|^2}{|\Sigma^{(k-r+1)}|^2} \\
&\gg |I_{k-r+1}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})| \\
&\frac{|I(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|^2}{|\Sigma^{(k-r+1)}|^2}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|C_{k,r+1}(P, \tilde{\Pi})| &\gtrsim \\
&\frac{|I(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|^{2(k-r+2)}}{|Q^{(k-r+1)}|^{k-r+1} |\Sigma^{(k-r+1)}|^{2(k-r+1)}} |D(P, \tilde{\Pi})| \prod_{j=k-r+2}^{k-1} |D(Q^{(j)} \Sigma^{(j)})|. \quad (81)
\end{aligned}$$

Before preceding, we must refine the set $C_{k-r+1,2}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})$ to a

slightly smaller set $C'_{k-r+1,2}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})$ defined as

$$C'_{k-r+1,2}(Q^{(k-r+1)}, \Sigma^{(k-r+1)}) := \left\{ \begin{array}{l} (\sigma_0, \sigma, q_0, \dots, q_{k-r+2}) \\ \in C_{k-r+1,2}(Q^{(k-r+1)}, \Sigma^{(k-r+1)}) \end{array} : \begin{array}{l} (\sigma_0, q_0) \\ \in D'(Q^{(k-r+1)}, \Sigma^{(k-r+1)}) \end{array} \right\}. \quad (82)$$

Making use of the uniform lower bound estimate from equation (53), and the fact that $|D'((Q^{(k-r+1)}, \Sigma^{(k-r+1)})| \geq \frac{1}{2}|D((Q^{(k-r+1)}, \Sigma^{(k-r+1)})|$, we have

$$|C'_{k-r+1,2}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})| \sim |C_{k-r+1,2}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})|.$$

We shall denote the elements of the set $C_{k,r+1}(P, \tilde{\Pi})$ which are cones of $(k-r+1, 2)$ -chains from $C_{k-r+1,2}(Q^{(k-r+1)}, \Sigma^{(k-r+1)})$ as $C'_{k,r+1}(P, \tilde{\Pi})$.

Next, it is necessary to simplify the expression on the right hand side this expression. Making use of equations (76) and (77), we have

$$\begin{aligned} & \frac{|I(Q^{(j_0-1)}, \Sigma^{(j_0-1)})|^{j_0(j_0-k+r)}}{|Q^{(j_0-1)}|^{(j_0-1)(j_0-k+r-1)+j_0-k+r-2}|\Sigma^{(j_0-1)}|^{(j_0-1)(j_0-k+r)}} \\ & |D(P, \tilde{\Pi})| \prod_{j=j_0}^{k-1} |D(Q^{(j)}, \Sigma^{(j)})| \\ & \approx \left(\frac{|\Sigma^{(j_0)}|}{|I(Q^{(j_0)}, \Sigma^{(j_0)})|} \right)^{j_0(j_0-k+r-1)-1} \left(\frac{|I(Q^{(j_0)}, \Sigma^{(j_0)})|^{2j_0+1}}{|Q^{(j_0-1)}|^{j_0}|\Sigma^{(j_0)}|^{2j_0-1}|D(Q^{(j_0)}, \Sigma^{(j_0)})|} \right)^{j_0-k+r} \\ & |D(P, \tilde{\Pi})| \prod_{j=j_0}^{k-1} |D(Q^{(j)}, \Sigma^{(j)})| \quad (83) \end{aligned}$$

holds when $k-r+2 \leq j_0 \leq k-1$. Furthermore, making use of the trivial estimate

$$|D(Q^{(j_0)}, \Sigma^{(j_0)})| \leq |Q^{(j_0)}||\Sigma^{(j_0)}|,$$

we arrive at an estimate of

$$\begin{aligned} |C'_{k,r+1}(P, \tilde{\Pi})| & \gtrsim \frac{|I(Q^{(j_0)}, \Sigma^{(j_0)})|^{(j_0+1)(j_0-k+r+1)}}{|Q^{(j_0)}|^{j_0(j_0-k+r-1)+j_0-k+r-1}|\Sigma^{(j_0)}|^{j_0(j_0-k+r+1)}} \\ & |D(P, \tilde{\Pi})| \prod_{j=j_0+1}^{k-1} |D(Q^{(j)}, \Sigma^{(j)})|. \quad (84) \end{aligned}$$

After repeating this procedure $r-2$ times, and making use of the trivial estimate $|D(P, \tilde{\Pi})| \leq |P||\tilde{\Pi}|$, we finally arrive at the desired conclusion

$$|C'_{k,r+1}(P, \tilde{\Pi})| \gtrsim \frac{|I(P, \tilde{\Pi})|^{(k+1)(r+1)}}{|P|^{kr+k-1}|\tilde{\Pi}|^{k(r+1)}} \quad (85)$$

Next, we shall compute a complementary upper bound for $|C'_{k,r+1}(P, \tilde{\Pi})|$ in two different ways (as was done for the case $r = 1$). First, choose a pair $(p_0, \pi_0) \in P \times \tilde{\Pi}$ such that $p_0 \notin \pi_0$; there are clearly at most $|P||\tilde{\Pi}|$ choices for this pair. By definition, the remaining $k+1$ points must lie inside π_0 , and they will uniquely define the remaining r k -planes once chosen. So, once the number of ways to choose the remaining points has been computed, we will have an upper bound for $|C'_{k,r+1}(P, \tilde{\Pi})|$. Since $|P \cap \pi_0| \approx \frac{|I|}{|\tilde{\Pi}|}$, and these points cannot cluster on low dimensional subspaces of π_0 , there are $\approx \left(\frac{|I|}{|\tilde{\Pi}|}\right)^{k-r}$ choices for the first $k-r$ points of the $(k, r+1)$ -chain. Next, assuming equation (70) holds, there are at most ~ 1 choices for the $(k-r)$ -plane found at the intersection of the $(r+1)$ -fold intersection of the k -planes in the $(k, r+1)$ chain. This means that we have at most $\sim |F|^{k-r}$ choices for the next point. Finally, there are $\approx \left(\frac{|I|}{|\tilde{\Pi}|}\right)^r$ choices for the remaining r points. These observations yield the bound

$$|C'_{k,r+1}(P, \tilde{\Pi})| \lesssim \frac{|P||I|^k|F|^{k-r}}{|\tilde{\Pi}|^{k-1}}. \quad (86)$$

Next, we make the same computation, only use the cruder estimate $|P \cap \pi_0| \leq F^k$ instead of $|P \cap \pi_0| \approx \frac{|I|}{|\tilde{\Pi}|}$. This method yields the bound

$$|C'_{k,r+1}(P, \tilde{\Pi})| \lesssim |P||\tilde{\Pi}||F|^{k^2+k-r}. \quad (87)$$

In order to finish the proof, we simply combine the upper and lower bounds we have now computed for $|C'_{k,r+1}(P, \tilde{\Pi})|$. Making use of the result from the previous computation, and the estimates (86) and (87), we have

$$|I(P, \tilde{\Pi})| \lesssim |P|^{\frac{r(k+1)}{(k+1)(r+1)-k}} |\tilde{\Pi}|^{\frac{(k+1)(r+1)-k-r}{(k+1)(r+1)-k}} |F|^{\frac{k-r}{(k+1)(r+1)-k}}, \quad (88)$$

and

$$|I(P, \tilde{\Pi})| \lesssim |P|^{\frac{r(k+1)}{(k+1)(r+1)}} |\tilde{\Pi}|^{\frac{(k+1)(r+1)-r}{(k+1)(r+1)-k}} |F|^{\frac{k^2+k-r}{(k+1)(r+1)}} \quad (89)$$

respectively.

As it turns out, our main estimate (25) is simply a convex combination of these two estimates. To be more specific, if we define the quantity $\alpha_r(k)$ as

$$\alpha_r(k) = \left(\frac{k(k+1)(r+1) - r(k^2 + 2k + 2)}{k^2 + 2k + 2} \right) \left(\frac{(k+1)(r+1) - k}{kr} \right), \quad (90)$$

then, one can compute that $0 \leq \alpha_r(k) \leq 1$ for appropriate values of k and r , and

$$\begin{aligned} |I(P, \tilde{\Pi})| &\lesssim (|P|^{\frac{r(k+1)}{(k+1)(r+1)-k}} |\tilde{\Pi}|^{\frac{(k+1)(r+1)-k-r}{(k+1)(r+1)-k}} |F|^{\frac{k-r}{(k+1)(r+1)-k}})^{\alpha_r(k)} \\ &\quad (|P|^{\frac{r(k+1)}{(k+1)(r+1)}} |\tilde{\Pi}|^{\frac{(k+1)(r+1)-r}{(k+1)(r+1)-k}} |F|^{\frac{k^2+k-r}{(k+1)(r+1)}})^{1-\alpha_r(k)} \\ &= |P|^{\frac{k(k+1)}{k^2+2k+2}} |\tilde{\Pi}|^{\frac{k^2+k+2}{k^2+2k+2}} |F|^{\frac{k(k+1)}{k^2+2k+2}}. \end{aligned}$$

□

6 Simplex Construction Part Three: The Upper Bound

Now that we have established a lower bound on the number of simplices appearing in the configuration of points and k -planes, we must find an upper bound. This task requires making a few observations about the set V_k . First of all, by construction, the pair of k -planes in any element of $v \in V_k$ spans some affine $(k+1)$ -dimensional space Λ_v . Since our original family of k -planes (and hence every refinement of that family) is direction separated, we have

$$|\{\pi \in \tilde{\Pi} : \pi \subset \Lambda_v\}| \lesssim |F|^k \quad (91)$$

for all $v \in V_k$. So, when constructing a simplex from some $v \in V_k$ by choosing the remaining k faces from Λ_v , we have $|F|^k$ choices for each face.

Before going ahead with this construction, however, one must observe that an element $v \in V_k$ is not merely a pair of k -planes; each such v comes equipped with a k -tuple of points on its “spine” (the intersection of the two k -planes). In order to manage this technicality, recall that (from the refined definition of \tilde{I}_k), the spine of any $v \in V_k$ must carry at least $\frac{|\tilde{I}|}{10|\tilde{\Pi}||F|}$ points from P . So, if we delete the points from each element of V_k to create the following set

$$V_{k,del} := \left\{ (\pi_0, \pi) \in \tilde{\Pi} \times \tilde{\Pi} : \begin{array}{l} \exists (p_1, \dots, p_k) \in P^k \text{ so that} \\ (\pi_0, \pi, p_1, \dots, p_k) \in V_k \end{array} \right\}, \quad (92)$$

we have the following bound:

$$|V_k| \gtrsim \left(\frac{|\tilde{I}|}{|\tilde{\Pi}||F|} \right)^k |V_{k,del}|. \quad (93)$$

Now that we have dealt with this possible over-counting, we may construct simplices from each element $v \in V_{del,k}$ by choosing the k remaining faces from Λ_v . This yields an upper bound in the number of simplices of

$$|S_k| \lesssim |V_k| \left(\frac{|\tilde{\Pi}||F|}{|\tilde{I}|} \right)^k |F|^{k^2}. \quad (94)$$

If we combine this with the lower bound obtained in Proposition 4.5, we arrive at the desired incidence bound:

$$|\tilde{I}| \lesssim |P|^{\frac{k(k+1)}{k^2+2k+2}} |\tilde{\Pi}|^{\frac{k^2+k+2}{k^2+2k+2}} |F|^{\frac{k(k+1)}{k^2+2k+2}}. \quad (95)$$

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